2.3. THE DETERMINANT OF A SQUARE MATRIX

DEFINITION. A **determinant** is a real number associated with every square matrix. The determinant of a square matrix A is denoted by $\det A$ or $|A|$, and defined inductively:

1) If $n = 1$, then $\det A = a_{11}$,
2) If $n > 1$, then

$$\det A = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = \sum_{k=1}^{n} (-1)^{1+k} a_{1k} M_{1k} =$$
$$= (-1)^{1+1} a_{11} M_{11} + (-1)^{1+2} a_{12} M_{12} + \cdots + (-1)^{1+n} a_{1n} M_{1n}$$

The right side of the above formula is called the **expansion of the determinant** of matrix A according to the first row. The (i,j) **minor**, denoted M_{ij}, is the determinant of the $(n-1) \times (n-1)$ matrix obtained from matrix A by deleting the ith row and the jth column.

The expression $C_{ij} = (-1)^{i+j} M_{ij}$ is called a **cofactor**. The **matrix of cofactors** A^{C} is the matrix found by replacing each element of a matrix A with its cofactor.

DEFINITION. A **non-singular matrix** is a square matrix in which the determinant does not equal zero ($\det A \neq 0$). A matrix is **singular** if and only if its determinant is zero.

THEOREM. The Laplace expansion (also called cofactor expansion)

$$\det A = \sum_{k=1}^{n} a_{ik} C_{ik} = \sum_{k=1}^{n} a_{kj} C_{kj}$$

The above theorem means that the determinant can be calculated by expanding it according to any chosen row or any chosen column.

For a 2×2 matrix, the determinant can be easily calculated using Leibniz formula:

$$\det \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = a_{11} \cdot a_{22} - a_{21} \cdot a_{12}$$

Example 1.

a) $\det \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix} = 2 \cdot 3 - 1 \cdot (-1) = 7$

b) $\det \begin{bmatrix} 0 & -1 \\ 2 & 3 \end{bmatrix} = 0 \cdot 3 - 2 \cdot (-1) = 2$

For 3×3 matrices and only such matrices, the determinant can be calculated using the **Sarrus' rule** (Sarrus' method). To calculate the determinant of matrix A using Sarrus' method we have to rewrite the two first rows below (or the two first columns) of the original matrix. Then we compute the products of the three diagonals from the left top to the right bottom and add them. From this value, we subtract the sum of the products of the three diagonals from the right top to the left bottom:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{matrix} a_{11}a_{22}a_{33} + a_{21}a_{32}a_{13} + a_{31}a_{12}a_{23} \\ -a_{13}a_{22}a_{31} - a_{23}a_{32}a_{11} - a_{33}a_{12}a_{21} \end{matrix}$$
$$\begin{matrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{matrix}$$

Example 2.
a)
$$\begin{vmatrix} 1 & 2 & 3 \\ -1 & 0 & 2 \\ 2 & 1 & 4 \end{vmatrix} = 1{\cdot}0{\cdot}4 + (-1){\cdot}1{\cdot}3 + 2{\cdot}2{\cdot}2 - 2{\cdot}0{\cdot}3 - 1{\cdot}1{\cdot}2 - (-1){\cdot}2{\cdot}4 = 11;$$
$$\begin{matrix} 1 & 2 & 3 \\ -1 & 0 & 2 \end{matrix}$$
b)
$$\begin{vmatrix} 1 & 2 & 1 \\ -1 & -4 & -2 \\ 2 & 1 & 4 \end{vmatrix} = \begin{matrix} 1 \cdot (-4) \cdot 4 + (-1) \cdot 1 \cdot 1 + 2 \cdot 2 \cdot (-2) \\ -2 \cdot (-4) \cdot 1 - 1 \cdot 1 \cdot (-2) - (-1) \cdot 2 \cdot 4 = -7 \end{matrix}$$
$$\begin{matrix} 1 & 2 & 1 \\ -1 & -4 & -2 \end{matrix}$$

For larger order determinants (determinants for matrices bigger than 3×3) it is necessary to use the Laplace expansion.

Example 3.
$$\begin{vmatrix} 1 & 2 & -1 & 0 \\ 0 & 3 & 1 & -2 \\ 4 & 1 & 5 & 2 \\ 2 & 1 & -1 & 0 \end{vmatrix} \underset{\substack{according \\ to\ the\ third \\ column}}{} $$
$$= (-1){\cdot}(-1)^{1+3} \cdot \begin{vmatrix} 0 & 3 & -2 \\ 4 & 1 & 2 \\ 2 & 1 & 0 \end{vmatrix} + 1{\cdot}(-1)^{2+3} \cdot \begin{vmatrix} 1 & 2 & 0 \\ 4 & 1 & 2 \\ 2 & 1 & 0 \end{vmatrix} +$$
$$+5 \cdot (-1)^{3+3} \cdot \begin{vmatrix} 1 & 2 & 0 \\ 0 & 3 & -2 \\ 2 & 1 & 0 \end{vmatrix} + (-1){\cdot}(-1)^{4+3} \cdot \begin{vmatrix} 1 & 2 & 0 \\ 0 & 3 & -2 \\ 4 & 1 & 2 \end{vmatrix} = -52$$

Note. Pick the row or column with the most zeros in it. In this case, that is the last column. For each element in the original matrix, its minor will be third-order determinants. We can calculate them using Sarrus' method or by expansion by minors - using three 2×2 determinants.

As the expansion of determinants with order n to determinants with order $n - 1$, $n - 2$, etc. is tedious and lengthy, to simplify it we use first step elementary row (column) operations.

DEFINITION. There are three types of **elementary row (column) operations**:

- **row (column) switching** – a row (column) within the matrix can be switched with another row (column) – the resulting determinant differs by a sign;
- **row (column) multiplication** – each element in a row (column) can be multiplied by a non-zero constant – the determinant is multiplied by the same non-zero constant;
- **row (column) addition** – a row (column) can be replaced by the sum of that row (column) and another row (column) can be multiplied by a non-zero constant – none of these operations change the value of the determinant.

Properties of determinants of matrices:

1) The expansion of the determinant according to any row is the same as according to any column.
2) The determinant of an identity matrix is 1 ($\det \boldsymbol{I}_n = 1$).
3) If A is a diagonal matrix, then $\det A = a_{11}a_{22} \cdot \ldots \cdot a_{nn}$.
4) If A is a triangular matrix (an upper triangular or lower triangular matrix), then $\det A = a_{11}a_{22} \cdot \ldots \cdot a_{nn}$.
5) $\det A^{\mathrm{T}} = \det A$.
6) If all the elements of a row (or column) are zeros, then the value of the determinant is zero ($\det A = 0$).
7) If two rows (or columns) of a determinant are identical the value of the determinant is zero ($\det A = 0$).
8) If any two rows (or two columns) of a determinant are switched, the value of the determinant is multiplied by -1.
9) If all elements of a row (or column) of the given determinant are multiplied by the same scalar k, the value of the new determinant is k times the given determinant. Therefore, if A is a square matrix of order n and K is any scalar then $\det(k \cdot A) = k^n \cdot \det A$.
10) $a_{k1} \cdot A_{i1} + a_{k2} \cdot A_{i2} + \cdots + a_{kn} \cdot A_{in} = \begin{cases} 0 & \text{for} \quad k \neq i \\ \det A & \text{for} \quad k = i \end{cases}$
11) Let A and B be two multipliable matrices - then $\det(A \cdot B) = \det A \cdot \det B$.
12) If a row (column) is replaced by the sum of that row (column) and another row (column) multiplied by a non-zero constant, the value of the determinant does not change.

In example 3, it is enough to add the third row to the second row and expand this determinant according to the fourth column:

$$\begin{vmatrix} 1 & 2 & -1 & 0 \\ 0 & 3 & 1 & -2 \\ 4 & 1 & 5 & 2 \\ 2 & 1 & -1 & 0 \end{vmatrix}_{R_3+R_2} = \begin{vmatrix} 1 & 2 & -1 & 0 \\ 4 & 4 & 6 & 0 \\ 4 & 1 & 5 & 2 \\ 2 & 1 & -1 & 0 \end{vmatrix} = 0 \cdot (-1)^{1+4} \cdot \begin{vmatrix} 4 & 4 & 6 \\ 4 & 1 & 5 \\ 2 & 1 & -1 \end{vmatrix} +$$

$$+0\cdot(-1)^{2+4}\cdot\begin{vmatrix}1&2&-1\\4&1&5\\2&1&-1\end{vmatrix}+2\cdot(-1)^{3+4}\cdot\begin{vmatrix}1&2&-1\\4&4&6\\2&1&-1\end{vmatrix}+0\cdot(-1)^{4+4}\cdot\begin{vmatrix}1&2&-1\\4&4&6\\4&1&5\end{vmatrix}=$$

$$=(-2)\cdot\begin{matrix}\begin{vmatrix}1&2&-1\\4&4&6\\2&1&-1\end{vmatrix}\\\begin{matrix}1&2&-1\\4&4&6\end{matrix}\end{matrix}=(-2)\cdot(-4-4+24+8-6+8)=-52$$

Example 4.

a) $$\begin{vmatrix}2&1&-1&2\\-1&2&1&4\\1&0&1&-1\\3&-1&4&0\end{vmatrix}_{\substack{R+(-2)R_1\\R_4+R_1}}=\begin{vmatrix}2&1&1&2\\-5&0&3&0\\1&0&1&-1\\5&0&3&2\end{vmatrix}_{\substack{\text{by second}\\ column}}=$$

$$=1\cdot(-1)^{1+2}\cdot\begin{matrix}\begin{vmatrix}-5&3&0\\1&1&-1\\5&3&2\end{vmatrix}\\\begin{matrix}-5&3&0\\1&1&-1\end{matrix}\end{matrix}=(-1)\cdot(-10+0-15-0-15-6)=46$$

b) $$\begin{vmatrix}2&1&-1&3\\1&0&-1&2\\2&2&1&3\\-1&2&3&-3\end{vmatrix}_{\substack{K_1+K_3\\K_1\cdot(-2)+K_4}}=\begin{vmatrix}2&1&1&-1\\1&0&0&0\\2&2&3&-1\\-1&2&2&-1\end{vmatrix}_{\substack{\text{by second}\\ row}}=$$

$$=1\cdot(-1)^{2+1}\cdot\begin{matrix}\begin{vmatrix}1&1&-1\\2&3&-1\\2&2&-1\end{vmatrix}\\\begin{matrix}1&1&-1\\2&3&-1\end{matrix}\end{matrix}=(-1)\cdot\left(-3-4-2+6+2+\dot{2}\right)=-1$$

Exercises

1. Find the value of the following determinants:

a) $\begin{vmatrix}\cos x&\sin x\\\sin x&\cos x\end{vmatrix}$, b) $\begin{vmatrix}\sin x&\sin 2x\\1&2\cos x\end{vmatrix}$, c) $\begin{vmatrix}3&0&1\\2&1&2\\0&0&1\end{vmatrix}$, d) $\begin{vmatrix}1&2&3\\3&2&1\\0&-2&1\end{vmatrix}$, e) $\begin{vmatrix}1&2&3\\2&4&-1\\5&-2&0\end{vmatrix}$,
f) $\begin{vmatrix}2&0&0\\3&4&0\\25&6&8\end{vmatrix}$, g) $\begin{vmatrix}1&2&3\\2&2&2\\3&4&5\end{vmatrix}$, h) $\begin{vmatrix}20&5&-60\\0&-4&10\\0&0&100\end{vmatrix}$, i) $\begin{vmatrix}1&2&3\\3&1&2\\2&3&1\end{vmatrix}$, j) $\begin{vmatrix}0&2&3\\2&0&2\\3&4&0\end{vmatrix}$,
k) $\begin{vmatrix}1&2&3\\4&5&6\\7&8&9\end{vmatrix}$.

2. Find the value of the following determinants:

$A=\begin{bmatrix}3&1&1&1\\0&-1&2&1\\1&1&3&1\\0&1&2&1\end{bmatrix}$, $B=\begin{bmatrix}1&-1&1&3\\1&1&1&2\\-1&-1&3&1\\2&1&1&1\end{bmatrix}$,