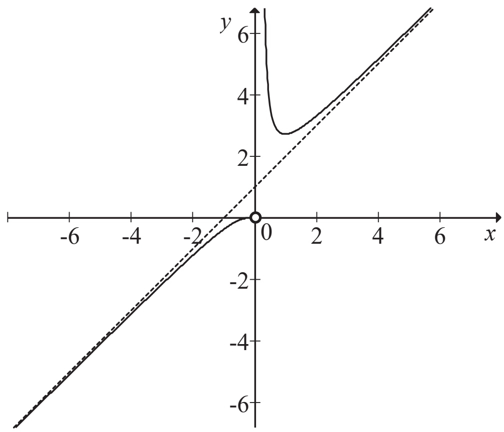

Fig. 5.14. The graph of $y = xe^{\frac{1}{x}}$

c) We find the properties of the function $y = x - 2\arctan x$ and sketch its graph according to the above steps.

1° The domain of the function is R.

2° We check the symmetry of the function:
$f(-x) = -x - 2\arctan(-x) = -x + 2\arctan x = -f(x)$. The function is odd.

3° To find the zeroes, the equation $x = 2\arctan x$ should be solved. It is easy to see that one of the solutions is $x = 0$ (then $y = 0$). The remaining zeroes can only be found using numerical methods. So, at this point, we limit ourselves to marking them on the graph of the function.

4° We find limits at infinity:
$\lim_{x\to-\infty}(x - 2\arctan x) = -\infty$ and $\lim_{x\to\infty}(x - 2\arctan x) = \infty$

5° The function has no vertical asymptotes. We check if oblique asymptotes exist:
$a = \lim_{x\to\pm\infty}\frac{x-2\arctan x}{x} = \lim_{x\to\pm\infty}\left(1 - \frac{2\arctan x}{x}\right) = 1$
$b_1 = \lim_{x\to-\infty}(x - 2\arctan x - x) = -2\lim_{x\to-\infty}\arctan x = -2\cdot\left(-\frac{\pi}{2}\right) = \pi$
$b_2 = \lim_{x\to+\infty}(x - 2\arctan x - x) = -2\lim_{x\to+\infty}\arctan x = -2\cdot\frac{\pi}{2} = -\pi$
The graph of the function has two oblique asymptotes:
left-hand (if $x \to -\infty$): $y = x + \pi$,
right-hand (if $x \to +\infty$): $y = x - \pi$.

6° We calculate the first derivative of the function: $y' = 1 - \frac{2}{1+x^2} = \frac{x^2-1}{1+x^2}$.
We find zeroes of the equation $y' = 0$: $\frac{x^2-1}{1+x^2} = 0$, then $x = -1 \vee x = 1$.
Now we analyse a sign of the first derivative. It depends only on the sign of the expression $x^2 - 1$: $y' < 0$ for $x \in (-1, 1)$ and $y' > 0$ for $x \in (-\infty, -1) \cup (1, \infty)$. So, the function has a local maximum at the point $x = -1$ equal to $y_{\max} = -1 + \frac{\pi}{2}$ and a local minimum at the point $x = 1$ equal to $y_{\min} = 1 - \frac{\pi}{2}$.

7° We calculate the second derivative of the function:

$$y'' = \frac{2x(1+x^2)-2x(x^2-1)}{(1+x^2)^2} = \frac{4x}{(1+x^2)^2}.$$

We solve the equation $y'' = 0$. We get $4x = 0$, so $x = 0$. We analyse the sign of the second derivative: $y'' < 0$ for $x < 0$ and $y'' > 0$ for $x > 0$. The second derivative changes its sign in the neighbourhood of the point $x = 0$. The point $P_{inf}(0,0)$ is the inflection point of the graph of the function.

8° The above results are summarised in the table below.

x	$(-\infty,-1)$	-1	$(-1,0)$	0	$(0,1)$	1	$(1,\infty)$
y'	+	0	–	–	–	0	+
y''	–	–	–	0	+	+	+
y	↗ $-\infty$	max $-1+\frac{\pi}{2}$	↘	P_{inf} 0	↘	min. $1-\frac{\pi}{2}$	↗ $+\infty$

9° We sketch the graph of the function:

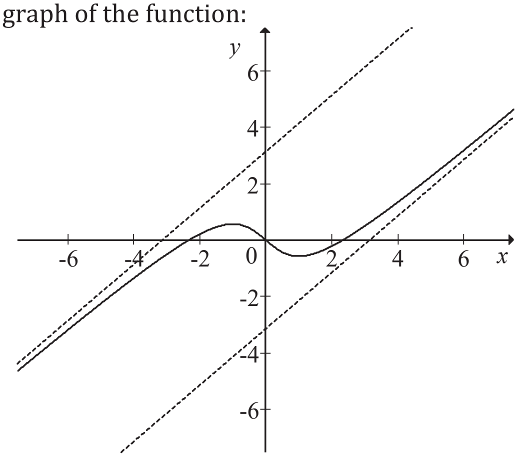

Fig. 5.15. The graph of $y = x - 2\arctan x$

d) We find the properties of the function $y = \frac{1}{\ln x}$ and sketch its graph according to the above steps.

1° The domain of the function is determined by the conditions: $\ln x \neq 0$ and $x > 0$, or $x \neq 1$ and $x > 0$. The domain of the function is the set $(0,1) \cup (1,\infty)$.

2° The function is neither periodic, even nor odd.

3° The graph of the function does not intersect the axes of the coordinate system, because $x \neq 0$ (because of the domain) and $y \neq 0$.

4° We calculate limits of the function at the endpoints of the domain:

$$\lim_{x\to 0^+} \frac{1}{\ln x} = 0, \quad \lim_{x\to 1^-} \frac{1}{\ln x} = -\infty, \quad \lim_{x\to 1^+} \frac{1}{\ln x} = +\infty, \quad \lim_{x\to +\infty} \frac{1}{\ln x} = 0$$